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Stationary Measures for the Periodic Euler Flow in Two Dimensions

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We construct for the Euler flow in two dimensions with periodic boundary conditions the Gibbsian measures given by the energy and the enstrophy integrals. We show that they are infinitesimally invariant under the Euler flow.

KEY WORDS: Stationary measures; periodic Euler; flow; two dimensions.

1. INTRODUCTION

The basic quantity of classical statistical mechanics describing large systems of particles in macroscopic equilibrium is the Gibbs measure on the state space of the system. Since for the mathematical study of equilibrium properties it is convenient to take the limit of infinite systems, equilibrium states for infinite systems of classical particles have been introduced as probability measures on the state space of the systems, namely as the limit points of Gibbsian measures for the corresponding finite systems (see, e.g., Refs. 1–4). In recent years these Gibbs measures for infinite systems of classical particles have also been used to provide the first results on the time evolution of such systems.[(4–17)] In fact it has been shown in a variety of cases that the solutions of Newton's equation of motion exist for almost all initial values with respect to the Gibbs measure in phase space[(4,14)] (see also for related work Refs. 5–17).

In a similar way as the evolution of a classical system of particles is given

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by the solution of the Newton equation as a mapping of the phase space into itself, the evolution of an incompressible fluid is described as a mapping of the space of velocities into itself given by the solutions of the Navier–Stokes (viscid case) or the Euler (inviscid case) equation of classical hydrodynamics. In the case of the Euler equation heuristic considerations based on the formal invariance of "the flat measure" in velocity space coupled with the existence of the invariance of the motion induced several authors to consider formal stationary measures of Gibbsian type (see, for example, the references in Ref. 18).

In this paper we construct Gibbsian measures $\mu_{\beta,\gamma}$ for the two-dimensional Euler flow with periodic boundary conditions on a square. We prove that the derivation corresponding to the infinitesimal flow given by the Euler equation defines a closable skew-symmetric operator $B \subset -B^*$ in $L_2(\mu_{\beta,\gamma})$ and that $\mu_{\beta,\gamma}$ is infinitesimally invariant under the Euler flow in the sense that $\int Bf \cdot d\mu_{\beta,\gamma} = 0$ for all f in the domain of B. This gives a first justification of the basic Ansatz for a "Gibbsian hydrodynamic" as a statistical mechanics in the function space of solutions of the Euler equation (analogous to the case of classical Gibbsian statistical mechanics, with the canonical equation of motion replaced by the Euler equation).

This result should also be seen in connection with the statistical approach to hydrodynamics (Kolmogorov, Onsager, Batchelor, and others), in particular with the functional approach to turbulence started by Hopf[19] and persued, for example, in Refs. 20–23 (see also for work on related equations, e.g., Refs. 24 and 25).

Our approach was strongly influenced by beautiful lectures by Gallavotti,[22] and we hope by our work to provide a justification for the ideas of Gallavotti and others (see Refs. 22 and 18) concerning the Euler flow.

It would be very interesting to know whether ($i \times$ times) the derivation B on $L_2(\mu_{\beta,\gamma})$ corresponding to the Euler equation is essentially self-adjoint. This would mean that there is a unique $\mu_{\beta,\gamma}$ measurable Euler flow.

As to the more detailed structure of this paper, in Section 2 we introduce the Euler equation and the associated conserved quantities, in particular the energy and the enstrophy. In Section 3 we define the standard normal distribution μ_γ given by the enstrophy integral. We then show the symmetry of iB in $L_2(d\mu_\gamma)$ and the infinitesimal invariance of μ_γ under the Euler flow. We also point out that the energy integral E is infinite almost surely with respect to μ_γ; however, a renormalization of it (subtraction of a suitable infinite constant) yields a function $:E:$ in $L_2(d\mu_\gamma)$. Then we show that the Gibbs measure $d\mu_{\beta,\gamma} = (\int e^{-\beta :E:}\, d\mu_\gamma)^{-1} e^{-\beta :E:}\, d\mu_\gamma$ are equivalent to μ_γ and infinitesimally invariant under the Euler flow, for all $\beta \geqslant 0$, $\gamma > 0$.

2. THE EULER EQUATION FOR AN INCOMPRESSIBLE FLUID ON A PERIODIC SQUARE

The Euler equation for an incompressible fluid in $\mathbb{R}^2$ is given by

$$\partial_t u = -(u \cdot \nabla)u - \nabla p, \qquad \operatorname{div} u = 0 \tag{2.1}$$

where $u = (u_1, u_2)$ is the velocity field of the fluid and $u \cdot \nabla = u_1 \partial_1 + u_2 \partial_2$ is the derivative in the direction u; $\nabla p = (\partial_1 p, \partial_2 p)$; and $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, $i = 1, 2$. Let $\nabla^\perp = (-\partial_2, \partial_1)$; then, since $\operatorname{div} u = 0$, we have that there is a function φ on $\mathbb{R}^2$ such that

$$u = \nabla^\perp \varphi = (-\partial_2 \varphi, \partial_1 \varphi) \tag{2.2}$$

The Euler equation (2.1) then takes the form

$$\partial_t \nabla^\perp \varphi = \sum_i (\nabla_i^\perp \varphi) \cdot \nabla_i \nabla^\perp \varphi - \nabla p \tag{2.3}$$

Since $\sum_i \nabla_i^\perp \cdot \nabla_i^\perp = \Delta$ and $\sum_i \nabla_i^\perp \nabla_i = 0$, where Δ is the Laplacian on $\mathbb{R}^2$, we may eliminate p from (2.3) and get

$$\partial_t \Delta \varphi = -\sum_{ij} \nabla_j^\perp [(\nabla_i^\perp \varphi) \cdot \nabla_i \nabla_j^\perp \varphi] \tag{2.4}$$

which may be written

$$\partial_t \Delta \varphi = -\sum_{ij} \nabla_j [(\nabla_i \varphi)\, \nabla_i^\perp \nabla_j \varphi] \tag{2.5}$$

using that $\sum_i (\nabla_i \nabla_j \varphi)(\nabla_i^\perp \nabla_j \varphi) = 0$. Thus

$$\partial_t \Delta \varphi = -\nabla \varphi \cdot \nabla^\perp \Delta \varphi \tag{2.6}$$

or equivalently

$$\partial_t \Delta \varphi = \nabla^\perp \varphi \cdot \nabla \Delta \varphi \tag{2.7}$$

Let now φ be a solution of (2.7) and set

$$f = -\partial_t \nabla^\perp \varphi - \sum_i (\nabla_i^\perp \varphi) \cdot \nabla_i \nabla^\perp \varphi \tag{2.8}$$

Since (2.4) is equivalent to (2.7), we get by the same rewriting as the one leading from (2.3) to (2.4) that $\nabla^\perp f = 0$. Since $\mathbb{R}^2$ is simply connected, this is equivalent to $f = \nabla p$ for some function p. With $u = \nabla^\perp \varphi$ we then see that u is a solution of the Euler equation (2.1). Hence we have the following theorem.

Theorem 2.1. u is a smooth solution of the Euler equation in $\mathbb{R}^2$

$$\partial_t u = -(u \cdot \nabla)u - \nabla p, \qquad \operatorname{div} u = 0$$

if and only if $u = \nabla^\perp \varphi$, $(u_1, u_2) = (-\partial_2 \varphi, \partial_1 \varphi)$, where φ is a smooth solution of the equation

$$\partial_t \Delta \varphi = \nabla^\perp \varphi \cdot \nabla \Delta \varphi$$

■

Following this theorem, we may consider (2.7) to be the Euler equation for an incompressible fluid on $\mathbb{R}^2$.

Let now Λ be the square $\Lambda = [0, 2\pi] \times [0, 2\pi]$, and we consider the equation (2.7) with periodic boundary conditions on Λ, i.e., we consider solutions φ_t of (2.7) such that $\varphi_t(0, y) = \varphi_t(2\pi, y)$ and $\varphi_t(x, 0) = \varphi_t(x, 2\pi)$. We call (2.7) with periodic boundary conditions on Λ *the Euler equation for an incompressible fluid on the periodic square* Λ.

We now define the *energy* E by

$$E = \tfrac{1}{2}\int_\Lambda u^2\, dx = -\tfrac{1}{2}\int_\Lambda \varphi\,\Delta\varphi\, dx \tag{2.9}$$

and the enstrophy

$$S = \tfrac{1}{2}\int (\operatorname{rot} u)^2\, dx = \tfrac{1}{2}\int (\Delta\varphi)^2\, dx \tag{2.10}$$

If $f \in C(\mathbb{R})$ we also introduce the notation

$$S_f = \int f(\operatorname{rot} u)\, dx = \int f(\Delta\varphi)\, dx \tag{2.11}$$

where $u = \nabla^\perp\varphi$ and φ is a smooth, periodic solution of (2.7) on $\Lambda = [0, 2\pi] \times [0, 2\pi]$. From (2.7) we then get

$$\frac{d}{dt}E = -\int_\Lambda \varphi\,\nabla^\perp\varphi\cdot\nabla\Delta\varphi\, dx = \int_\Lambda \nabla\varphi\cdot\nabla^\perp\varphi\,\Delta\varphi\, dx = 0 \tag{2.12}$$

since $\nabla\varphi\cdot\nabla^\perp\varphi = 0$, and also

$$\frac{d}{dt}S = \int_\Lambda \Delta\varphi\,\nabla^\perp\varphi\,\nabla\Delta\varphi\, dx = -\int_\Lambda \nabla^\perp\Delta\varphi\cdot\nabla\Delta\varphi\, dx = 0 \tag{2.13}$$

since $\nabla^\perp\Delta\varphi\cdot\nabla\Delta\varphi = 0$.

If $f \in C^2(\mathbb{R})$ we get

$$\frac{d}{dt}S_f = \int f'(\Delta\varphi)\,\nabla^\perp\varphi\cdot\nabla\Delta\varphi\, dx = -\int f''(\Delta\varphi)\,\nabla^\perp\Delta\varphi\cdot\nabla\Delta\varphi\, dx = 0 \tag{2.14}$$

For $f \in C(\mathbb{R})$ we get $(d/dt)S_f = 0$ by approximating f uniformly by C^2 functions. Hence we have the following theorem:

Theorem 2.2. Let φ be a smooth solution of the Euler equation for an incompressible fluid in the periodic square $\Lambda = [0, 2\pi] \times [0, 2\pi]$, i.e.,

$$\partial_t\Delta\varphi = \nabla^\perp\varphi\cdot\nabla\Delta\varphi$$

such that

$$\varphi(0, y, t) = \varphi(2\pi, y, t) \qquad \text{and} \qquad \varphi(x, 0, t) = \varphi(x, 2\pi, t)$$

Then $E = \frac{1}{2}\int_\Lambda (\nabla^\perp\varphi)^2\,dx$, $S = \frac{1}{2}\int_\Lambda (\Delta\varphi)^2\,dx$, and, for $f \in C(\mathbb{R})$, $S_f = \int_\Lambda f(\Delta\varphi(x))\,dx$ are constants of motion, i.e., independent of t. ∎

Let us now expand $\varphi(x, t)$ in Fourier series

$$\varphi(x, t) = \frac{1}{2\pi} \sum_{\substack{k\in Z^2 \\ k\neq 0}} \omega_k(t) e^{ikx} \tag{2.15}$$

where $\omega_{-k} = \bar{\omega}_k$ since φ is real and we have chosen φ such that $\int_\Lambda \varphi\,dx = 0$. This we may always do since an additive constant obviously drops out of Eq. (2.7) as well as from the velocity field $u = \nabla^\perp\varphi$. Equation (2.7) now takes the form

$$k^2 \frac{d}{dt}\omega_k = -\sum_{h+h'=k} (h^\perp\cdot h')(h')^2\omega_h\omega_{h'} \tag{2.16}$$

where $h^\perp = (-h_2, h_1)$. Since $h^\perp\cdot h' = h_1h_2' - h_2h_1'$ is antisymmetric in h and h', (2.16) may also be written

$$k^2 \frac{d}{dt}\omega_k = \frac{1}{2}\sum_{h+h'=k} (h^\perp h')[h^2 - (h')^2]\omega_h\omega_{h'} \tag{2.17}$$

The energy E and the enstrophy S are given respectively by

$$E = \frac{1}{2}\sum_k k^2|\omega_k|^2 \qquad \text{and} \qquad S = \frac{1}{2}\sum_k k^4|\omega_k|^2 \tag{2.18}$$

If we introduce the functions $B_k(\omega)$ defined on the space of sequences $\{\omega_k\}$, $k\in Z^2$, $k \neq 0$, such that $\omega_k = \bar{\omega}_{-k}$ by

$$B_k(\omega) = \frac{1}{2k^2}\sum_{h+h'=k} (h^\perp h')[h^2 - (h')^2]\omega_h\omega_{h'} \tag{2.19}$$

then (2.17) takes the form

$$\frac{d}{dt}\omega_k = B_k(\omega), \qquad k\in Z^2, \quad k\neq 0 \tag{2.20}$$

Since for $h + h' = k$, $(h^\perp\cdot h') = (h\cdot k)$ and $h^2 - (h')^2 = -k^2 + 2(h\cdot k)$, we have

$$B_k(\omega) = \sum_h \left[\frac{1}{k^2}(h^\perp\cdot k)(h\cdot k) - \frac{1}{2}(h^\perp\cdot k)\right]\omega_h\cdot\omega_{k-h} \tag{2.21}$$

From (2.19) we see that

$$\partial B_k/\partial\omega_k = 0, \qquad k\in Z^2, \quad k\neq 0 \tag{2.22}$$

3. GIBBS MEASURES FOR THE EULER FLOW

For $k \in Z^2$ we write $k > 0$ iff $k = (k_1, k_2)$ with $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$. Let H_S be the complex Hilbert space of sequences $\{\omega_k\}_{k>0}$ with the inner product

$$(\omega, \omega)_S = S(\omega) = \sum_{k>0} k^4 |\omega_k|^2 \tag{3.1}$$

Let $H_{\gamma S}$ be the same Hilbert space with the inner product $\gamma \cdot (\omega, \omega)_S = \gamma S(\omega)$ for $\gamma > 0$. Let $d\mu_\gamma$ be the standard normal distribution associated with the complex Hilbert space $H_{\gamma S}$ and let E_γ be the expectation with respect to $d\mu_\gamma$. We then have that

$$E_\gamma(e^{i\gamma(u,\omega)_S}) = \int e^{i\gamma(u,\omega)_S}\, d\mu_\gamma(\omega) = e^{-\gamma(u,u)_S/2} \tag{3.2}$$

for any $u \in H_S$. Here μ_γ is a probability measure on a compact space X_S which contains H_S as a dense set of measure zero. The probability space (X_S, μ_γ) may for instance be constructed as follows.

Let $\mu_\gamma{}^k$ be the probability measure on $\mathbb{C}$, the complex numbers, given by

$$d\mu_\gamma{}^k(z) = (2\pi/\gamma k^4)^{-1} \exp(-\tfrac{1}{2}\gamma k^4 |z|^2)\, dx\, dy \tag{3.3}$$

for any $k \in Z^2$, $k > 0$, where $z = x + iy$, and let $\overline{\mathbb{C}}$ be the one-point compactification of $\mathbb{C}$. Then

$$(X_S, \mu_\gamma) = \sum_{\substack{k \in Z^2 \\ k>0}} (\overline{\mathbb{C}}, \mu_\gamma{}^k) \tag{3.4}$$

is a realization of the standard normal distribution on $H_{\gamma S}$. Remark that ω_h and $\omega_{h'}$ are independent normally distributed for $h \neq h'$, and their expectation is zero, i.e., $E_\gamma(\omega_h) = 0$, while their covariance is given by

$$E_\gamma(\omega_h \omega_{h'}) = 0 \tag{3.5}$$

and

$$E_\gamma(\overline{\omega}_h \omega_{h'}) = 2\delta_{hh'} \gamma^{-1} h^{-4} \tag{3.6}$$

Let now

$$B_k{}^n(\omega) = \sum_{h^2 \leqslant n} \left[\frac{1}{k^2}(h^\perp \cdot k)(h \cdot k) - \frac{1}{2}(h^\perp \cdot k)\right] \omega_h \omega_{k-h} \tag{3.7}$$

$B_k{}^n(\omega)$ is a measurable function on (X_S, μ_γ). Moreover,

$$E_\gamma(|B_k{}^n|^2) = \sum_{\substack{h^2 \leqslant n \\ (h')^2 \leqslant n}} \left[\frac{1}{k^2}(h^\perp \cdot k)(h \cdot k) - \frac{1}{2}(h^\perp \cdot k)\right]$$

$$\times \left\{\frac{1}{k^2}[(h')^\perp \cdot k](h' \cdot k) - \frac{1}{2}[(h')^\perp \cdot k]\right\} E(\overline{\omega}_h \overline{\omega}_{k-h} \omega_{h'} \omega_{k-h'}) \tag{3.8}$$

But

$$E(\overline{\omega}_h \overline{\omega}_{k-h} \omega_{h'} \omega_{k-h'}) = 4\delta_{hh'} \gamma^{-2} h^{-4} (k-h)^{-4} + 4\delta_{h,k-h'} \gamma^{-2} h^{-4} (k-h)^{-4} \quad (3.9)$$

Hence

$$E_\gamma(|B_k^n|^2) \leqslant 8 \sum_{\substack{h \neq 0 \\ h \neq k}} \frac{1}{k^4} (h^\perp \cdot k)^2 (h \cdot k)^2 \gamma^{-2} h^{-4} (k-h)^{-4} \quad (3.10)$$

or

$$E_\gamma(|B_k^n|^2) \leqslant 8\gamma^{-2} \sum_{h \neq k} (k-h)^{-4} \quad (3.11)$$

In the same way we get for $m < n$ that

$$E_\gamma(|B_k^n - B_k^m|^2) \leqslant 8\gamma^{-2} \sum_{\substack{h \neq k \\ h^2 > m}} (k-h)^{-4} \quad (3.12)$$

This proves that B_k^n converges in $L_2(d\mu_\gamma)$ to $B_k(\omega)$ as $n \to \infty$. Therefore $B_k(\omega)$ is μ_γ-measurable and in fact in $L_2(d\mu_\gamma)$.

Let now $FC^1(H_S) = FC^1$ be the space of differentiable functions on H_S depending only on a finite number of coordinates ω_k. Obviously FC^1 is dense in $L_2(d\mu_\gamma)$. Let B be the vector field

$$B = \sum B_k(\omega) \frac{\partial}{\partial \omega_k} \quad (3.13)$$

Then B is defined on FC^1 and maps FC^1 into $L_2(d\mu_\gamma)$. From (2.22) we have that B is divergence-free,

$$\operatorname{div} B = \sum_k \frac{\partial B_k}{\partial \omega_k} = 0 \quad (3.14)$$

Now since B is divergence-free and S is invariant under any solution of the equation

$$\frac{d}{dt} \omega_k = B_k(\omega), \qquad k \in Z^2, \quad k \neq 0 \quad (3.15)$$

one may expect that μ_γ is an invariant measure for the Euler flow. Since we have not constructed solutions of (3.15) for μ_γ-almost all initial conditions ω, the question of whether μ_γ is invariant is, of course, premature. However, another relevant question is whether μ_γ is infinitesimally invariant in the sense that

$$\int Bf \, d\mu_\gamma = 0 \quad (3.16)$$

for all $f \in D(B) = FC^1$, i.e., whether $B^* \cdot 1 = 0$, B^* being the adjoint in $L_2(d\mu_\gamma)$. To compute B^* we need to compute $(\partial/\partial\omega_k)^*$. Now if e_k is the kth coordinate direction, then

$$\int \bar{f} \frac{\partial g}{\partial \omega_k} \, d\mu_\gamma = \frac{d}{dt} \int \bar{f}(\omega) g(\omega + te_k) \, d\mu_\gamma(\omega) \Big|_{t=0} \tag{3.17}$$

for any $f, g \in FC^1$.

It is easily seen from (3.3) and (3.4) that

$$\mu_\gamma(\omega - te_k) = [\exp(-\tfrac{1}{2}\gamma k^4 t^2) \exp(-\gamma k^4 t \bar{\omega}_k)] \mu_\gamma(\omega) \tag{3.18}$$

Therefore

$$\int \bar{f} \frac{\partial g}{\partial \omega_k} \, d\mu_\gamma(\omega) = -\int \overline{\left(\frac{\partial}{\partial \bar{\omega}_k} f\right)} g \, d\mu_\gamma(\omega) - \gamma k^4 \int \bar{\omega}_k \bar{f} g \, d\mu_\gamma(\omega) \tag{3.19}$$

Thus

$$\left(\frac{\partial}{\partial \omega_k}\right)^* = \frac{\partial}{\partial \omega_{-k}} - \gamma k^4 \omega_k \tag{3.20}$$

From this we get, since $\bar{B}_k(\omega) = B_{-k}(\omega)$,

$$\begin{aligned} B^* f &= -\sum_k \left(\frac{\partial}{\partial \omega_k} + \gamma k^4 \omega_k\right) B_{-k} f \\ &= -\left(\sum_k \frac{\partial B_k}{\partial \omega_k} + \gamma \sum_k k^4 \bar{\omega}_k B_k\right) f - \sum_k B_k \frac{\partial f}{\partial \omega_k} \end{aligned} \tag{3.21}$$

By (3.14), $\sum_k (\partial B_k/\partial \omega_k) = 0$, and from (3.1) we have that

$$\frac{d}{dt} S = \sum B_k \frac{\partial S}{\partial \omega_k} = \sum_k k^4 \bar{\omega}_k B_k \tag{3.22}$$

and by Theorem 2.2, $dS/dt = 0$, so that for $f \in D(B) = FC^1$ we have

$$B^* f = -Bf \tag{3.23}$$

i.e., $B^* \supset -B$. In particular we get that $B^*1 = -B1 = 0$, so that μ_γ is infinitesimally invariant. Hence we have the following theorem:

Theorem 3.1. Let $u(x, t) = (i/2\pi) \sum_{k \neq 0} k^\perp \omega_k(t) e^{ikx}$, $\bar{\omega}_k = \omega_{-k}$, be the Fourier expansion for $u(x, t)$. Then the Euler equation (2.1) is equivalent to the equation

$$\frac{d}{dt} \omega_k(t) = B_k(\omega), \qquad k \in Z^2, \quad k \neq 0$$

with

$$B_k(\omega) = \frac{1}{2k^2} \sum_{h + h' = k} (h^\perp \cdot h')[h^2 - (h')^2] \omega_h \omega_{h'}$$

The energy and enstrophy are given by $E = \frac{1}{2}\sum_k k^2|\omega_k|^2$ and $S = \sum k^4|\omega_k|^2$. Let $H_{\gamma S}$ be the Hilbert space with square norm given by γS for $\gamma > 0$ and μ_γ the standard normal distribution associated with $H_{\gamma S}$; then $B_k(\omega) \in L_2(d\mu_\gamma)$ for any $\gamma > 0$. Moreover, the vector field $B = \sum_k B_k\, \partial/\partial\omega_k$ maps $FC^1(H_S)$ into $L_2(d\mu_\gamma)$ and thus defines a densely defined operator on $L_2(d\mu_\gamma)$. Moreover, iB is symmetric in the sense that $B^* \supset -B$ and therefore B is closable. The measure μ_γ is infinitesimally invariant under the Euler flow in the sense that

$$\int Bf\, d\mu_\gamma = \sum_k \int B_k \frac{\partial f}{\partial \omega_k}\, d\mu_\gamma = 0$$

for any $f \in FC^1$. ∎

Set now

$$E_n = \frac{1}{2} \sum_{k^2 \leqslant n} k^2 |\omega_k|^2 \tag{3.24}$$

Then $E_n(\omega) \nearrow E(\omega)$ as $n \to \infty$. From (3.6) we get that

$$\int E_n(\omega)\, d\mu_\gamma(\omega) = \gamma^{-1} \sum_{k^2 \leqslant n} \frac{1}{k^2} \tag{3.25}$$

which tends to infinity as $n \to \infty$. Therefore $E(\omega)$ is not μ_γ-integrable. On the other hand, setting

$$:E: \equiv \frac{1}{2} \sum_k \left(k^2 |\omega_k|^2 - \frac{2}{\gamma k^2} \right) \tag{3.26}$$

we have that $:E:(\omega) \in L_2(d\mu_\gamma)$, in fact

$$\int (:E:)^2\, d\mu_\gamma = \gamma^{-2} \sum_k k^{-4} \tag{3.27}$$

This implies that $:E:(\omega)$ is finite for μ_γ-almost all ω, which in turn implies that $E(\omega)$ is infinite for μ_γ-almost all ω. Hence with respect to the stationary measure μ_γ the energy E is infinite with probability one. It is well known from the theory of normal distributions μ associated with a Hilbert space H that if E is a quadratic function on H given by a Hilbert Schmidt operator, then $e^{-\beta :E:}$ is in $L_1(d\mu)$ for any $\beta > 0$. Since $\sum k^{-4} < \infty$ we have that E is given by a Hilbert Schmidt operator on $H_{\gamma S}$, hence that

$$e^{-\beta :E:} \in L_1(d\mu_\gamma) \tag{3.28}$$

for any positive β and γ. Therefore

$$d\mu_{\beta,\gamma} = \left(\int e^{-\beta :E:}\, d\mu_\gamma \right)^{-1} e^{-\beta :E:}\, d\mu_\gamma \tag{3.29}$$

is a probability measure which is equivalent to μ_γ. It follows from well-known results about normal distributions that μ_γ and $\mu_{\gamma'}$ are disjoint if $\gamma \neq \gamma'$. We call $\mu_{\beta,\gamma}$ for positive γ and nonnegative β the *Gibbs measure* for the Euler flow. One easily proves that $B^*{:}E{:} = 0$, from which it follows that $B^* e^{-\beta :E:} = 0$, which gives us that $\mu_{\beta,\gamma}$ are infinitesimally invariant. Thus we have the following result:

Theorem 3.2. The energy $E = \frac{1}{2}\sum_k k^2|\omega_k|^2$ is infinite for μ_γ-almost all ω. If we define

$$:E: = \frac{1}{2}\sum_k \left(k^2|\omega_k|^2 - \frac{2}{\gamma k^2}\right)$$

then $:E:$ is in $L_2(d\mu_\gamma)$ and thus finite μ_γ-almost everywhere. Moreover, $e^{-\beta :E:} \in L_1(d\mu_\gamma)$ for any $\beta \geqslant 0$ and any $\gamma > 0$. The Gibbs measures

$$d\mu_{\beta,\gamma} = \left(\int e^{-\beta :E:}\, d\mu_\gamma\right)^{-1} e^{-\beta :E:}\, d\mu_\gamma$$

are equivalent to μ_γ and infinitesimally invariant in the sense that, for any $f \in FC^1$, $\int Bf\, d\mu_{\beta,\gamma} = 0$. Moreover, $:E:$ is an infinitesimally invariant function in the sense that $:E: \in D(B^*)$ and $B^*{:}E{:} = 0$. The measures μ_γ and $\mu_{\gamma'}$ have disjoint support for $\gamma \neq \gamma'$.

NOTE ADDED IN PROOF

After submission of this paper we received a preprint "Equilibrium states for the two-dimensional incompressible Euler fluid" by C. Boldreghini and S. Frigio (Camerino and Roma) which contains related results, obtained independently.

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